

Practice - 11

Double Integrals in Polar Coordinates

8 701 273 47 85 – What's Up

Multiple integrals:

a) Double integrals

- 1) Iterated Integrals
- 2) Double Integrals over General Regions
- 3) Double Integrals in Polar Coordinates

b) Triple integrals

- 1) Triple Integrals in Cylindrical Coordinates
- 2) Triple Integrals in Spherical Coordinates

Definite integrals

Before starting on double integrals let's do a quick review of the definition of **definite integrals** for **functions of single variables**.

First, when working with the integral, $\int_a^b f(x) dx$

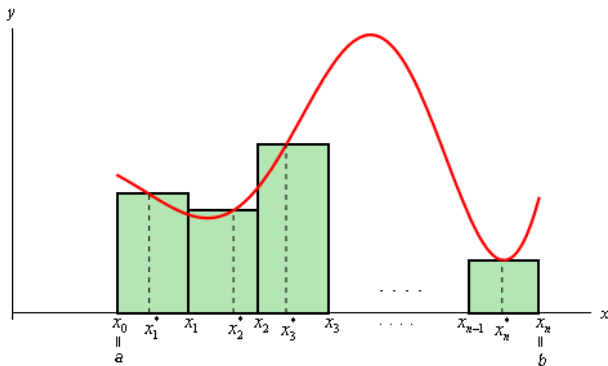
we think of **x's** as coming from the interval **$a \leq x \leq b$** .

For these integrals we can say that we are integrating over the interval **$a \leq x \leq b$** .

Definite integrals

Now, when we derived the definition of the **definite integral** we first thought of this as **an area problem**.

We first asked **what the area under the curve was** and to do this we broke up the interval $a \leq x \leq b$ into n subintervals of width Δx and choose a point, x_i^* , from each interval as shown below,



Definite integrals

Each of the rectangles has height of $f(x_i^*)$ and we could then use the area of each of these rectangles to approximate the area as follows.

$$A \approx f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_i^*) \Delta x + \cdots + f(x_n^*) \Delta x$$

To get the **exact area** we then took the limit as n goes to infinity and this was also **the definition of the definite integral**.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Double integrals

Now we want to integrate a function of two variables, $f(x,y)$.

With functions of one variable we integrated over an interval (i.e. a **one-dimensional space**) and so it makes some sense then that when integrating a function of two variables we will integrate over a region of $\mathbb{R}^2$ (**two-dimensional space**).

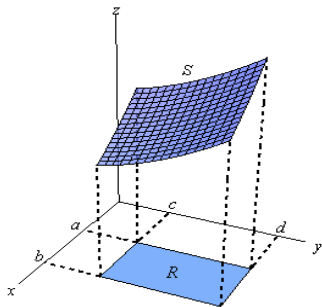
Double integrals

We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows, $R = [a, b] \times [c, d]$

This means that the ranges for x and y are $a \leq x \leq b$ and $c \leq y \leq d$.

Also, we will initially assume that $f(x,y) \geq 0$ although this doesn't really have to be the case.

Let's start out with the graph of the surface S given by graphing $f(x,y)$ over the rectangle R .



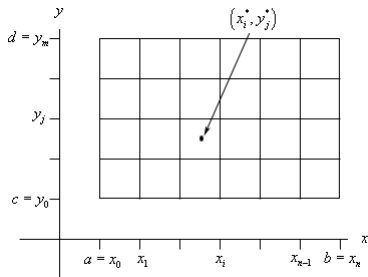
Double integrals

Let's first ask what the volume of the region under S (and above the xy -plane of course) is.

We will approximate the volume much as we approximated the area above.

We will first divide up $a \leq x \leq b$ into n subintervals and divide up $c \leq y \leq d$ into m subintervals. This will divide up R into a series of smaller rectangles and from each of these we will choose a point (x_i^*, y_j^*) .

Here is a sketch of this set up.

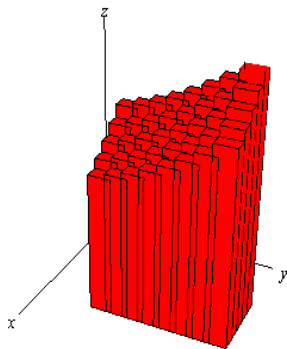


Double integrals

Now, **over each of these smaller rectangles** we will construct a box whose height is given by $f(x_i^*, y_j^*)$.

Here is a sketch of that.

Each of the rectangles has a base area of ΔA and a height of $f(x_i^*, y_j^*)$ so the volume of each of these boxes is $f(x_i^*, y_j^*) \Delta A$.



The volume under the surface S is then approximately,

$$V \approx \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

Double integrals

$$V \approx \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

We will have a **double sum** since we will need to add up volumes in both the **x** and **y** directions.

To get a better estimation of the volume we will take **n** and **m** **larger** and **larger** and to get the **exact volume** we will need to take the **limit** as both **n** and **m** go to infinity.

$$V = \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

Here is the official **definition of a double integral** of a **function of two variables** over a rectangular region R as well as the notation that we'll use for it.

$$\iint_R f(x, y) dA = \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

Double integrals

Note that the differential is dA instead of the dx and dy that we're used to seeing.

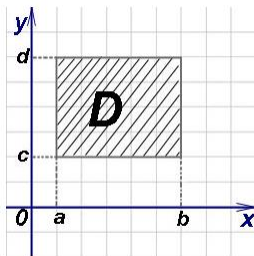
As indicated above one interpretation of the double integral of $f(x,y)$ over the rectangle R is the volume under the function $f(x,y)$ (and above the xy -plane). Or,

$$\text{Volume} = \iint_R f(x,y) dA$$

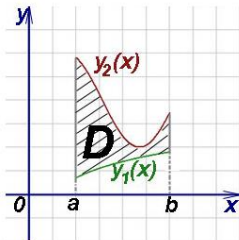
Double integrals

a) Double integrals

1) Iterated Integrals



2) Double Integrals over General Regions



Theorem 1 (Fubini's Theorem):

If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$ then,

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

These integrals are called **iterated integrals**.

Double integrals

- 1) **Area**: if $z=f(x,y)=1$, then the double integrals gives the area of region D .

$$\text{Area of } D = \iint_D dA$$

- 2) **Volume**: the integral is equal to volume under the surface $z=f(x,y) \neq 1$ above the region D .

$$\text{Volume} = \iint_D f(x, y) dA$$

Ways of computing a double integral over a rectangle.

$$\iint_R f(x, y) \, dA = \int_a^b \left[\int_c^d f(x, y) \, dy \right] dx$$

We will compute the double integral by first computing

$$\int_c^d f(x, y) \, dy$$

and we compute this by holding **x constant** and **integrating with respect to y** as if this were a single integral.

This will give **a function involving only x's** which we can in turn integrate.

Ex1: (a) $\iint_R 6xy^2 dA,$ $R = [2, 4] \times [1, 2]$

It doesn't matter which variable we integrate with respect to first, we will get the same answer regardless of the order of integration.

Ex1: Solution 1.

$$\iint_R 6xy^2 dA = \int_2^4 \int_1^2 6xy^2 dy dx$$

In this case we will integrate with respect to y first.

$$\begin{aligned}\iint_R 6xy^2 dA &= \int_2^4 (2xy^3) \Big|_1^2 dx \\ &= \int_2^4 (16x - 2x) dx = \\ &= \int_2^4 14x dx = 7x^2 \Big|_2^4 = 84\end{aligned}$$

Ex1: Solution 2.
$$\iint_R 6xy^2 dA = \int_2^4 \int_1^2 6xy^2 dy dx$$

In this case we'll integrate with respect to x first and then y .

$$\begin{aligned}\iint_R 6xy^2 dA &= \int_1^2 \int_2^4 6xy^2 dx dy \\ &= \int_1^2 (3x^2y^2)|_2^4 dy \\ &= \int_1^2 36y^2 dy \\ &= 12y^3|_1^2 \\ &= 84\end{aligned}$$

Sure enough the same answer as the first solution.

So, remember that we can do the integration in any order.

Ex2: (b) $\iint_R (2x - 4y^3) dA, \quad R = [-5, 4] \times [0, 3]$

For this integral we'll integrate with respect to **y** first.

$$\begin{aligned} \iint_R (2x - 4y^3) dA &= \int_{-5}^4 \int_0^3 (2x - 4y^3) dy dx \\ &= \int_{-5}^4 (2xy - y^4) \Big|_0^3 dx \\ &= \int_{-5}^4 (6x - 81) dx \\ &= (3x^2 - 81x) \Big|_{-5}^4 \\ &= -756 \end{aligned}$$

Substitutions in Double integrals !!!

Ex3: (e) $\iint_R x e^{xy} dA, \quad R = [-1, 2] \times [0, 1]$

Now, while we can technically integrate with respect to either variable first sometimes one way is significantly easier than the other way.

In this case it will be significantly easier to integrate with respect to y first as we will see.

$$\iint_R x e^{xy} dA = \int_{-1}^2 \int_0^1 x e^{xy} dy dx$$

The y integration can be done with the quick **substitution**,

$$u = xy \quad du = x dy$$

which gives

Substitutions in Double integrals !!!

Ex3: which gives

$$\begin{aligned}\iint_R x e^{xy} dA &= \int_{-1}^2 e^{xy} \Big|_0^1 dx \\&= \int_{-1}^2 (e^x - 1) dx \\&= (e^x - x) \Big|_{-1}^2 \\&= e^2 - 2 - (e^{-1} + 1) \\&= e^2 - e^{-1} - 3\end{aligned}$$

Substitutions in Double integrals !!!

Ex3: So, not too bad of an integral there provided you get the substitution.
Now let's see what would happen if we had integrated with respect to x first.

$$\iint_R x e^{xy} dA = \int_0^1 \int_{-1}^2 x e^{xy} dx dy$$

In order to do this we would have to use
integration by parts as follows,

$$\begin{array}{ll} u = x & dv = e^{xy} dx \\ du = dx & v = \frac{1}{y} e^{xy} \end{array}$$

The integral is then,

$$\begin{aligned} \iint_R x e^{xy} dA &= \int_0^1 \left(\frac{x}{y} e^{xy} - \int \frac{1}{y} e^{xy} dx \right) \Big|_{-1}^2 dy \\ &= \int_0^1 \left(\frac{x}{y} e^{xy} - \frac{1}{y^2} e^{xy} \right) \Big|_{-1}^2 dy \\ &= \int_0^1 \left(\frac{2}{y} e^{2y} - \frac{1}{y^2} e^{2y} \right) - \left(-\frac{1}{y} e^{-y} - \frac{1}{y^2} e^{-y} \right) dy \end{aligned}$$

We're not even going to continue here as these are very difficult, if not impossible, integrals to do.

Theorem 2:

If $f(x, y) = g(x) h(y)$ and we are integrating
over the rectangle $R = [a, b] \times [c, d]$ then,

$$\begin{aligned}\iint_R f(x, y) \, dA &= \iint_R g(x) h(y) \, dA = \\ &= \left(\int_a^b g(x) \, dx \right) \left(\int_c^d h(y) \, dy \right)\end{aligned}$$

Ex4: Evaluate

$$\iint_R x \cos^2(y) \, dA, \quad R = [-2, 3] \times \left[0, \frac{\pi}{2}\right].$$

Since the integrand is a function of x times a function of y we can use the fact.

$$\begin{aligned} \iint_R x \cos^2(y) \, dA &= \left(\int_{-2}^3 x \, dx \right) \left(\int_0^{\frac{\pi}{2}} \cos^2(y) \, dy \right) \\ &= \left(\frac{1}{2} x^2 \right) \Big|_{-2}^3 \left(\frac{1}{2} \int_0^{\frac{\pi}{2}} 1 + \cos(2y) \, dy \right) \\ &= \left(\frac{5}{2} \right) \left(\frac{1}{2} \left(y + \frac{1}{2} \sin(2y) \right) \Big|_0^{\frac{\pi}{2}} \right) \\ &= \frac{5\pi}{8} \end{aligned}$$

Task-1: Evaluate

$$\iint_D xy \, dx \, dy,$$

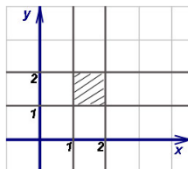
where $D = \{(x, y) \mid 1 \leq x \leq 2; 1 \leq y \leq 2\}$.

Task-1: Evaluate $\iint_D xy \, dx \, dy$, where $D = \{(x, y) \mid 1 \leq x \leq 2; 1 \leq y \leq 2\}$.

Solution:

Step-1:

$$\int_1^2 xy \, dx = y \cdot \left. \frac{x^2}{2} \right|_1^2 = 2y - \frac{y}{2}$$



Step-2:

$$\begin{aligned} \int_1^2 \left(2y - \frac{y}{2} \right) dy &= \left(\frac{2y^2}{2} - \frac{y^2}{4} \right) \bigg|_1^2 = \frac{3}{4} y^2 \bigg|_1^2 = \\ &= \frac{3}{4} \cdot (4 - 1) = \frac{9}{4}. \end{aligned}$$

Task-2: Evaluate

$$\iint_D (4 - x^2 - y^2) dx dy,$$

if the region D is bounded by straight lines:

$$x = 0, \quad x = 1, \quad y = 0, \quad y = \frac{3}{2}.$$

Task-2: Evaluate $\iint_D (4 - x^2 - y^2) dx dy$,

if the region D is bounded by straight lines: $x = 0, x = 1, y = 0, y = \frac{3}{2}$

Solution:

Step-1:
$$\int_0^{\frac{3}{2}} dy \int_0^1 (4 - x^2 - y^2) dx$$

Step-2:
$$\begin{aligned} \int_0^1 (4 - x^2 - y^2) dx &= 4x - \frac{x^3}{3} - y^2 x \Big|_0^1 = \\ &= 4 - \frac{1}{3} - y^2. \end{aligned}$$

Step-3:
$$\begin{aligned} \int_0^{\frac{3}{2}} \left(4 - \frac{1}{3} - y^2 \right) dy &= \left(4y - \frac{1}{3}y - \frac{y^3}{3} \right) \Big|_0^{\frac{3}{2}} = \\ &= \frac{35}{8}. \end{aligned}$$

Double Integrals Over General Regions

In the previous section we looked at double integrals over rectangular regions.

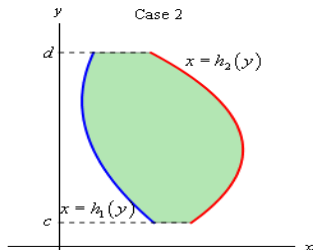
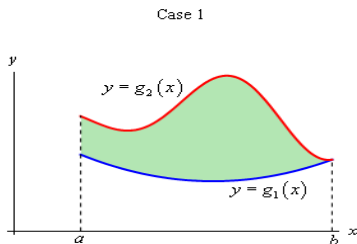
The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\iint_D f(x, y) \, dA$$

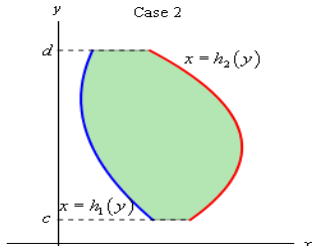
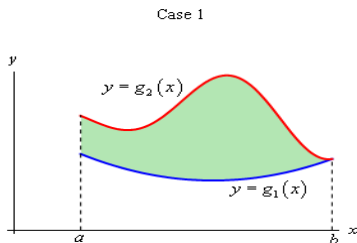
where D is any region.

There are two types of regions that we need to look at.

Here is a sketch of both of them.



Double Integrals Over General Regions



We will often use **set builder notation** to describe these regions.

Here is the definition for the region in

Case 1

$$D = \{(x, y) \mid a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\}$$

Case 2

$$D = \{(x, y) \mid h_1(y) \leq x \leq h_2(y), \ c \leq y \leq d\}$$

Double Integrals Over General Regions

The double integral for both of these cases are defined in terms of iterated integrals as follows.

In Case 1 where $D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ the integral is defined to be,

$$\iint_D f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx$$

In Case 2 where $D = \{(x, y) \mid h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$ the integral is defined to be,

$$\iint_D f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy$$

Double Integrals Over General Regions

Here are **some properties** of the double integral.

Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

Properties:

$$1. \iint_D f(x, y) + g(x, y) \, dA = \iint_D f(x, y) \, dA + \iint_D g(x, y) \, dA$$

$$2. \iint_D cf(x, y) \, dA = c \iint_D f(x, y) \, dA, \text{ where } c \text{ is any constant.}$$

3. If the region D can be split into two separate regions D_1 and D_2 then the integral can be written as

$$\iint_D f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA$$

Double Integrals Over General Regions

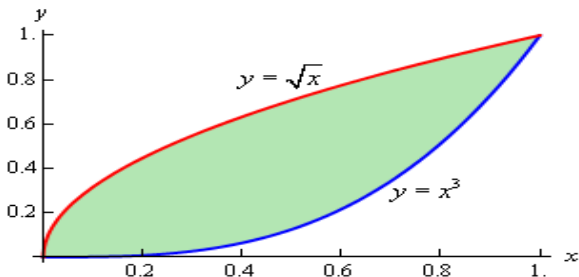
Ex5:

$$\iint_D (4xy - y^3) dA$$

D is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

In this case we need to determine the two inequalities for x and y that we need to do the integral.

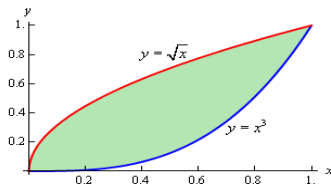
The best way to do this is the graph the two curves.



Double Integrals Over General Regions

Ex5: So, from the sketch
we can see that that two inequalities are,

$$0 \leq x \leq 1 \quad x^3 \leq y \leq \sqrt{x}$$



We can now do the integral,

$$\begin{aligned} \iint_D (4xy - y^3) dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} (4xy - y^3) dy dx \\ &= \int_0^1 \left(2xy^2 - \frac{1}{4}y^4 \right) \Big|_{x^3}^{\sqrt{x}} dx = \\ &= \int_0^1 \left(\frac{7}{4}x^2 - 2x^7 + \frac{1}{4}x^{12} \right) dx = \\ &= \left(\frac{7}{12}x^3 - \frac{1}{4}x^8 + \frac{1}{52}x^{13} \right) \Big|_0^1 = \frac{55}{156} \end{aligned}$$

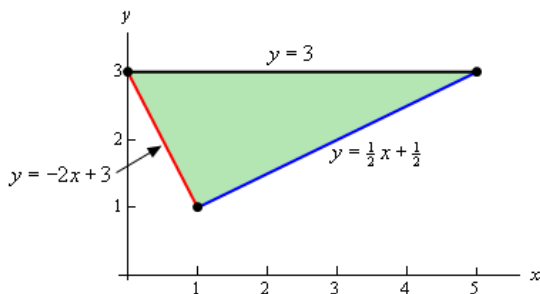
Double Integrals Over General Regions

Ex6:

$$\iint_D 6x^2 - 40y \, dA,$$

D is the triangle with vertices $(0, 3)$, $(1, 1)$, and $(5, 3)$.

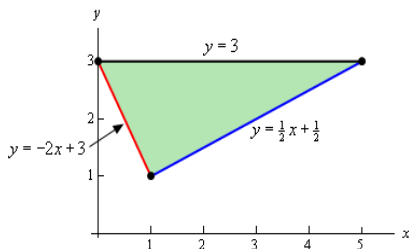
We got even less information about the region this time.
Let's start this off by sketching the triangle.



Double Integrals Over General Regions

Ex6:

If we use functions of x ,
as shown in the image
we will have to break
the region up into two different pieces
since the lower function is different
depending upon the value of x .



In this case the region would be given by $D = D_1 \cup D_2$,
where,

$$D_1 = \{(x, y) \mid 0 \leq x \leq 1, -2x + 3 \leq y \leq 3\}$$

$$D_2 = \left\{ (x, y) \mid 1 \leq x \leq 5, \frac{1}{2}x + \frac{1}{2} \leq y \leq 3 \right\}$$

Double Integrals Over General Regions

Ex6: 1) A(0;3) , B(1;1).

$$\frac{x - x_a}{x_b - x_a} = \frac{y - y_a}{y_b - y_a}$$

$$\frac{x - 0}{1 - 0} = \frac{y - 3}{1 - 3}$$

$$\frac{x}{1} = \frac{y - 3}{-2}$$

$$y = -2x + 3$$

2) B(1;1), C(5;3)

$$\frac{x - x_a}{x_b - x_a} = \frac{y - y_a}{y_b - y_a}$$

$$\frac{x - 1}{5 - 1} = \frac{y - 1}{3 - 1}$$

$$\frac{x - 1}{4} = \frac{y - 1}{2}$$

$$y = 0.5x + 0.5$$

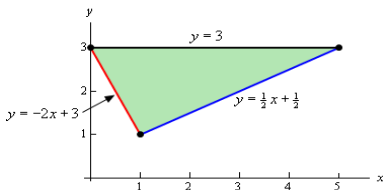
3) A(0;3) , B(5;3).

$$\frac{x - x_a}{x_b - x_a} = \frac{y - y_a}{y_b - y_a}$$

$$\frac{x - 0}{5 - 0} = \frac{y - 3}{3 - 3}$$

$$y_b - y_a = 0,$$

$$y = 3$$



Double Integrals Over General Regions

Ex6: Solution 1:

$$\begin{aligned}\iint_D 6x^2 - 40y \, dA &= \iint_{D_1} 6x^2 - 40y \, dA + \iint_{D_2} 6x^2 - 40y \, dA \\&= \int_0^1 \int_{-2x+3}^3 6x^2 - 40y \, dy \, dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^3 6x^2 - 40y \, dy \, dx \\&= \int_0^1 (6x^2y - 20y^2) \Big|_{-2x+3}^3 \, dx + \int_1^5 (6x^2y - 20y^2) \Big|_{\frac{1}{2}x+\frac{1}{2}}^3 \, dx \\&= \int_0^1 12x^3 - 180 + 20(3-2x)^2 \, dx + \int_1^5 -3x^3 + 15x^2 - 180 + 20\left(\frac{1}{2}x + \frac{1}{2}\right)^2 \, dx \\&= \left(3x^4 - 180x - \frac{10}{3}(3-2x)^3\right) \Big|_0^1 + \left(-\frac{3}{4}x^4 + 5x^3 - 180x + \frac{40}{3}\left(\frac{1}{2}x + \frac{1}{2}\right)^3\right) \Big|_1^5 \\&= -\frac{935}{3}\end{aligned}$$

Double Integrals Over General Regions

Ex6: Solution 2: Note the $\cup$ is the “union” symbol and just means that D is the region we get by combining the two regions. If we do this then we'll need to do two separate integrals, one for each of the regions. To avoid this we could turn things around and solve the two equations for x to get,

$$y = -2x + 3 \quad \Rightarrow \quad x = -\frac{1}{2}y + \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2} \quad \Rightarrow \quad x = 2y - 1$$

If we do this we can notice that the same function is always on the right and the same function is always on the left and so the region is,

$$D = \left\{ (x, y) \mid -\frac{1}{2}y + \frac{3}{2} \leq x \leq 2y - 1, \quad 1 \leq y \leq 3 \right\}$$

Writing the region in this form means doing a single integral instead of the two integrals we'd have to do otherwise.

Double Integrals Over General Regions

Ex6: Solution 2:

This solution will be a lot less work since we are only going to do a single integral.

$$\begin{aligned}\iint_D 6x^2 - 40y \, dA &= \int_1^3 \int_{-\frac{1}{2}y + \frac{3}{2}}^{2y-1} 6x^2 - 40y \, dx \, dy \\&= \int_1^3 (2x^3 - 40xy) \Big|_{-\frac{1}{2}y + \frac{3}{2}}^{2y-1} dy \\&= \int_1^3 100y - 100y^2 + 2(2y-1)^3 - 2\left(-\frac{1}{2}y + \frac{3}{2}\right)^3 dy \\&= \left(50y^2 - \frac{100}{3}y^3 + \frac{1}{4}(2y-1)^4 + \left(-\frac{1}{2}y + \frac{3}{2}\right)^4\right) \Big|_1^3 \\&= -\frac{935}{3}\end{aligned}$$

Double Integrals Over General Regions

Task-3: Evaluate

$$\iint_D (x^2 + xy + 2y^2) dx dy,$$

where $D = \{(x, y) \mid 0 \leq x \leq 1; 0 \leq y \leq 1-x\}$.

Double Integrals Over General Regions

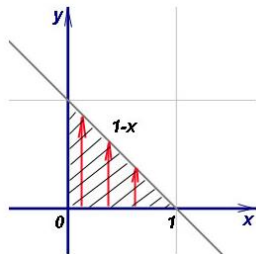
Task-3: Evaluate $\iint_D (x^2 + xy + 2y^2) dx dy$,

where $D = \{(x, y) | 0 \leq x \leq 1; 0 \leq y \leq 1-x\}$.

Solution:

Step-1:
$$\int_0^1 dx \int_0^{1-x} (x^2 + xy + 2y^2) dy$$

Step-2:
$$\begin{aligned} & \int_0^{1-x} (x^2 + xy + 2y^2) dy = \\ & = x^2(1-x) + \frac{1}{2}x(1-x)^2 + \frac{2}{3}(1-x)^3. \end{aligned}$$



Double Integrals Over General Regions

Step-3:

$$\int_0^1 x^2(1-x)dx + \frac{1}{2} \int_0^1 x(1-x)^2 dx + \frac{2}{3} \int_0^1 (1-x)^3 dx.$$

1)

$$\begin{aligned} \int_0^1 x^2(1-x)dx &= \int_0^1 (x^2 - x^3)dx = \\ &= \frac{1}{3}x^3 \Big|_0^1 - \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{12}. \end{aligned}$$

2)

$$\begin{aligned} \frac{1}{2} \int_0^1 x(1-x)^2 dx &= \frac{1}{2} \int_0^1 \left(\frac{1}{2}x - x^2 + \frac{1}{2}x^3 \right) dx = \\ &= \frac{1}{4}x^2 \Big|_0^1 - \frac{1}{3}x^3 \Big|_0^1 + \frac{1}{8}x^4 \Big|_0^1 = \frac{1}{24}. \end{aligned}$$

3)

$$\begin{aligned} \frac{2}{3} \int_0^1 (1-x)^3 dx &= \int_0^1 \left(\frac{2}{3} - 2x + 2x^2 - \frac{2}{3}x^3 \right) dx = \\ &= \frac{2}{3}x^3 \Big|_0^1 - \frac{1}{6}x^4 \Big|_0^1 - x^2 \Big|_0^1 + \frac{2}{3}x \Big|_0^1 = \frac{1}{6}. \end{aligned}$$

Step-4:

$$\frac{1}{12} + \frac{1}{24} + \frac{1}{6} = \frac{7}{24}.$$

Double Integrals Over General Regions

Task-4: Evaluate $\iint_D dx \, dy$,

where $D = \{(x, y) \mid 0 \leq x \leq 2; -4 + x^2 \leq y \leq 4 - x^2\}$.

Double Integrals Over General Regions

Task-4: Evaluate $\iint_D dx \, dy$,

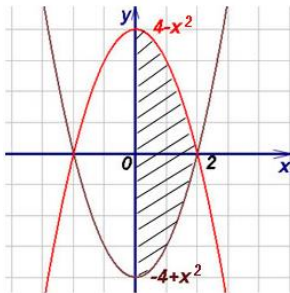
where $D = \{(x, y) \mid 0 \leq x \leq 2; -4 + x^2 \leq y \leq 4 - x^2\}$

Solution:

Step-1:
$$\int_0^2 dx \int_{-4+x^2}^{4-x^2} dy$$

Step-2:

$$\int_{-4+x^2}^{4-x^2} dy = 4 - x^2 - (-4 + x^2) = 8 - 2x^2$$



Double Integrals Over General Regions

Step-3:

$$\begin{aligned}\int_0^2 (8 - 2x^2) dx &= \left(8x - \frac{2}{3}x^3 \right) \bigg|_0^2 = \\ &= 16 - \frac{16}{3} = \frac{32}{3}.\end{aligned}$$

Double Integrals Over General Regions

Task-5: Evaluate

$$\iint_D xy \, dx dy,$$

if the region D is bounded by the lines: $y = x, xy = 1, y = 2.$

Double Integrals Over General Regions

Task-5: Evaluate $\iint_D xy \, dx dy$,

if the region D is bounded by the lines: $y = x$, $xy = 1$, $y = 2$.

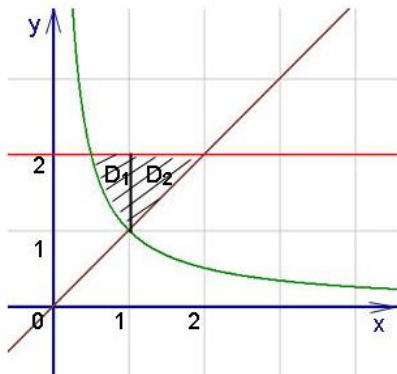
Solution:

Step-1:

a) $y = x$, $A(1;1)$
 $xy = 1$

b) $y = x$, $B(2;2)$
 $y = 2$.

c) $xy = 1$, $C(1/2;2)$
 $y = 2$.



Double Integrals Over General Regions

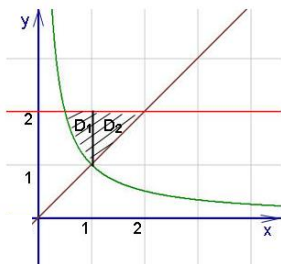
Step-2:

$$\iint_D xy \, dx dy = \iint_{D_1} xy \, dx dy + \iint_{D_2} xy \, dx dy =$$

$$= \int_{\frac{1}{2}}^1 dx \int_{\frac{1}{x}}^2 xy \, dx + \int_1^2 dx \int_x^2 xy \, dx =$$

$$= \int_{\frac{1}{2}}^1 x \frac{y^2}{2} \bigg|_{\frac{1}{x}}^2 dx + \int_1^2 x \frac{y^2}{2} \bigg|_x^2 dx + \int_{\frac{1}{2}}^1 \left(2x - \frac{1}{2x} \right) dx + \int_1^2 \left(2x - \frac{x^3}{2} \right) dx =$$

$$= \left(x^2 - \frac{1}{2} \ln x \right) \bigg|_{\frac{1}{2}}^1 + \left(x^2 - \frac{1}{8} x^4 \right) \bigg|_1^2 = 1 + \frac{1}{2} \ln \frac{1}{2} - \frac{1}{4} + 4 - 2 + \frac{1}{8} = \frac{15}{8} - \frac{1}{2} \ln 2.$$



Double Integrals In Polar Coordinates

However, in every case we've seen to this point the **region D** could be easily **described** in terms of simple functions **in Cartesian coordinates**.

In this section we want to look at **some regions** that are much easier to **describe** in terms of **polar coordinates**.

For instance, we might have a region that is a **disk, ring, or a portion of a disk or ring**.

In these cases, using Cartesian coordinates could be somewhat cumbersome.

Double Integrals In Polar Coordinates

For instance, let's suppose we wanted to do the following integral,

$$\iint_D f(x, y) \, dA, \quad D \text{ is the disk of radius 2}$$

To this we would have to determine a set of inequalities for x and y that describe this region. These would be,

$$\begin{aligned} -2 &\leq x \leq 2 \\ -\sqrt{4-x^2} &\leq y \leq \sqrt{4-x^2} \end{aligned}$$

With these limits the integral would become,

$$\iint_D f(x, y) \, dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y) \, dy \, dx$$

Due to the limits on the inner integral this is liable to be an unpleasant integral to compute.

Double Integrals In Polar Coordinates

$$\iint_D f(x, y) \, dA,$$

D is the disk of radius 2

However, a disk of radius 2 can be defined in polar coordinates by the following inequalities,

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 2$$

These are very simple limits and, in fact, are constant limits of integration which almost always makes integrals somewhat easier.

The problem is that we can't just convert the dx and the dy into a dr and a $d\theta$.

In computing double integrals to this point we have been using the fact that $dA=dx dy$ and this really does require Cartesian coordinates to use.

Double Integrals In Polar Coordinates

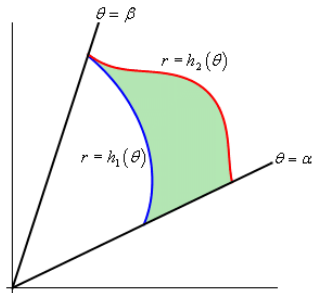
Once we've moved into **polar coordinates** $dA \neq drd\theta$ and so we're going to need to **determine** just what dA is under **polar coordinates**.

A **general region** in terms of **polar coordinates**.

Here is a sketch of some region using polar coordinates.

So, our general region will be defined by inequalities,

$$\alpha \leq \theta \leq \beta$$
$$h_1(\theta) \leq r \leq h_2(\theta)$$



Double Integrals In Polar Coordinates

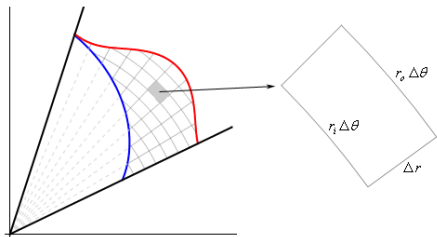
Now, to find dA let's redo the figure above as follows,

As shown, we'll break up the region into a mesh of radial lines and arcs. Now, if we pull one of the pieces of the mesh out as shown **we have something** that is almost, but **not quite a rectangle**.

The **area** of this piece is ΔA .

The two sides of this piece both have length $\Delta r = r_o - r_i$ where r_o is the radius of the **outer arc** and r_i is the radius of the **inner arc**.

Basic geometry then tells us that the length of the inner edge is $r_i \Delta \theta$ while the length of the out edge is $r_o \Delta \theta$ where $\Delta \theta$ is the angle between the two radial lines that form the sides of this piece.



Double Integrals In Polar Coordinates

Now, let's assume that we've taken the mesh **so small** that we can assume $r_i \approx r_o = r$ that and with this assumption we can also assume that **our piece is close enough to a rectangle** that we can also then assume that,

$$\Delta A \approx r \Delta \theta \Delta r$$

Also, if we assume that the mesh is small enough then we can also assume that,

$$dA \approx \Delta A \quad d\theta \approx \Delta \theta \quad dr \approx \Delta r$$

With these assumptions we then get $dA \approx r dr d\theta$.

In fact, as the mesh size gets smaller and smaller the formula above becomes more and more accurate and so we can say that,

$$\underline{dx \, dy} = dA = r \, dr \, d\theta$$

$$dA \neq dr \, d\theta.$$

Double Integrals In Polar Coordinates

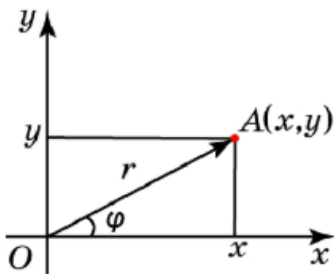
Now, if we're going to be **converting** an **integral in Cartesian coordinates** **into** an **integral in polar coordinates** we are going to have to make sure that we've also converted all the **x's** and **y's** into **polar coordinates** as well. To do this we'll need to remember the following conversion formulas,

$$x = r \cos \theta \qquad y = r \sin \theta \qquad r^2 = x^2 + y^2$$

We are now ready to write down a formula for the **double integral** in terms of **polar coordinates**.

$$\iint_D f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta$$

Polar Coordinates (r , θ)



$$r = \sqrt{x^2 + y^2},$$

$$\cos \phi = \frac{x}{\sqrt{x^2 + y^2}}, \sin \phi = \frac{y}{\sqrt{x^2 + y^2}}.$$

$$x = r \cos \phi,$$

$$y = r \sin \phi.$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r^2 = x^2 + y^2$$

Jacobian of the polar coordinates

Def: The **Jacobian** of the polar coordinates transformation

$$x = r \cos \theta, \quad y = r \sin \theta$$

is the 2×2 determinant

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

Here

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos(\theta), & \frac{\partial y}{\partial r} &= \sin(\theta), \\ \frac{\partial x}{\partial \theta} &= -r \sin(\theta), & \frac{\partial y}{\partial \theta} &= r \cos(\theta), \end{aligned}$$

This explains why there's an r factor in polar integrals! The area element $dA = dx dy$ is not equal to $dr d\theta$. Instead, dA is equal to $r dr d\theta$.

Ex7: Evaluate the following integral by converting them into polar coordinates.

$$(a) \iint_D 2xy \, dA,$$

D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

First let's get D in terms of polar coordinates. The circle of radius 2 is given by $r=2$ and the circle of radius 5 is given by $r=5$.

We want the region between the two circles, so we will have the following inequality for r : $2 \leq r \leq 5$

Also, since we only want the portion that is in the first quadrant we get the following range of θ 's:

$$0 \leq \theta \leq \frac{\pi}{2}$$

Double Integrals In Polar Coordinates

$$x = r \cos \theta \quad y = r \sin \theta$$

Ex7:

$$(a) \iint_D 2xy \, dA, \quad \begin{array}{l} 2 \leq r \leq 5 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{array} \quad \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \end{array}$$

Now that we've got these we can do the integral.

$$\iint_D 2xy \, dA = \int_0^{\frac{\pi}{2}} \int_2^5 2(r \cos \theta)(r \sin \theta) r \, dr \, d\theta$$

$$\begin{aligned} \iint_D 2xy \, dA &= \int_0^{\frac{\pi}{2}} \int_2^5 r^3 \sin(2\theta) \, dr \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left. \frac{1}{4} r^4 \sin(2\theta) \right|_2^5 d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{609}{4} \sin(2\theta) \, d\theta \\ &= -\frac{609}{8} \cos(2\theta) \Big|_0^{\frac{\pi}{2}} = \frac{609}{4} \end{aligned}$$

Double Integrals In Polar Coordinates

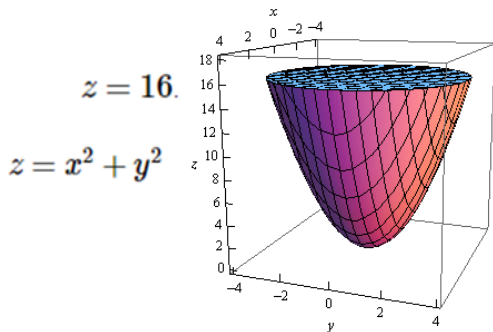
$$x = r \cos \theta$$

$$y = r \sin \theta$$

Ex8: Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

Note: $z = f(x, y)$ so $f(x, y) = 16$, $f(x, y) = x^2 + y^2$

Let's start this example off with a quick sketch of the region.



Ex8: The formula $V = \iint_D f(x, y) dA$

finds the **volume** **under** the function **f(x,y)** and we're actually after the volume that is above a function.

First, notice that $V = \iint_D 16 dA$ will be the volume under $z=16$ (of course

we'll need to determine D eventually) while $V = \iint_D x^2 + y^2 dA$

is the volume under $z = x^2 + y^2$, using the same D .

The volume that we're after is really the difference between these two or,

$$V = \iint_D 16 dA - \iint_D x^2 + y^2 dA = \iint_D 16 - (x^2 + y^2) dA$$

Double Integrals In Polar Coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Ex8: Now all that we need to do is to **determine** the region **D** and then convert everything over to **polar coordinates**.

Determining the region D in this case is not too bad.

If we were to look straight down the **z**-axis onto the region we would see a **circle of radius 4** centered at the origin.

This is because **the top of the region**, where the elliptic paraboloid intersects the plane, is the widest part of the region.

We know the z coordinate at the intersection so, setting $z=16$ in the equation of the paraboloid gives, $16 = x^2 + y^2$ which is the equation of **a circle of radius 4 centered at the origin**.

Here are the inequalities for the region and the function we'll be integrating in terms of polar coordinates.

$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 4 \quad z = 16 - r^2$$

Ex8: The volume is then,

$$\begin{aligned} V &= \iint_D 16 - (x^2 + y^2) \, dA \\ &= \int_0^{2\pi} \int_0^4 r (16 - r^2) \, dr \, d\theta \\ &= \int_0^{2\pi} \left(8r^2 - \frac{1}{4}r^4 \right) \Big|_0^4 \, d\theta \\ &= \int_0^{2\pi} 64 \, d\theta \\ &= 128\pi \end{aligned}$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 4$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$dA = r \, dr \, d\theta$$

Task-6: Calculate the double integral using polar coordinates:

$$\int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{-\sqrt{2-x^2}}^0 \frac{xy}{x^2+y^2} dy$$

Solution:

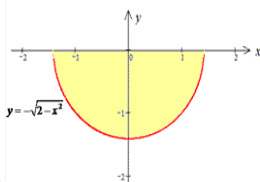
Step-1: We reduce the functions of the integration limits

$$-\sqrt{2} \leq x \leq \sqrt{2}, \quad -\sqrt{2-x^2} \leq y \leq 0$$

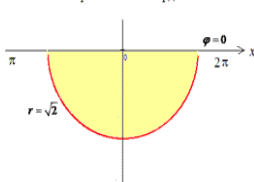
to the canonical form

$$y = -\sqrt{2-x^2} \rightarrow y^2 = 2-x^2 \rightarrow x^2 + y^2 = (\sqrt{2})^2.$$

Прямокутна система координат

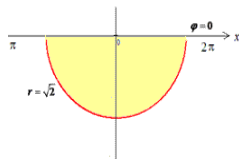


Полярна система координат



$$\int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{-\sqrt{2-x^2}}^0 \frac{xy}{x^2+y^2} dy$$

Полярна система координат



Step-2: We turn the integrand into polar coordinates:

$$\frac{xy}{x^2+y^2} = \frac{r \cos \varphi \cdot r \sin \varphi}{r^2 \cos^2 \varphi + r^2 \sin^2 \varphi} = \frac{r^2 \sin 2\varphi}{2r^2} = \frac{\sin 2\varphi}{2}.$$

Step-3: $0 \leq r \leq \sqrt{2}, \pi \leq \varphi \leq 2\pi.$

Step-4:

$$\begin{aligned} \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{-\sqrt{2-x^2}}^0 \frac{xy}{x^2+y^2} dy &= \frac{1}{2} \int_{\pi}^{2\pi} d\varphi \int_0^{\sqrt{2}} r \sin 2\varphi r dr = \frac{1}{4} \int_{\pi}^{2\pi} \sin 2\varphi \cdot r^2 \Big|_0^{\sqrt{2}} d\varphi = \\ &= \frac{1}{2} \int_{\pi}^{2\pi} \sin 2\varphi d\varphi = -\frac{1}{4} \cos 2\varphi \Big|_{\pi}^{2\pi} = -\frac{1}{4} (\cos 4\pi - \cos 2\pi) = 0. \end{aligned}$$

Task-7: Calculate the double integral using polar coordinates:

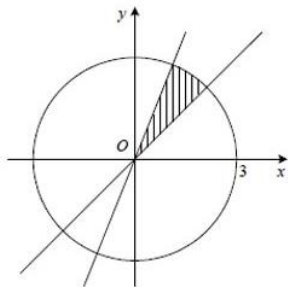
$$\iint_D \sqrt{9 - x^2 - y^2} dx dy,$$

if the region D is bounded by the lines: $x^2 + y^2 = 9$, $y = x$, $y = \sqrt{3}x$.

Solution:

Step-1:

$$\begin{aligned}\sqrt{9 - x^2 - y^2} &= \sqrt{9 - (r \cos \varphi)^2 - (r \sin \varphi)^2} = \\ &= \sqrt{9 - r^2 (\cos^2 \varphi + \sin^2 \varphi)} = \sqrt{9 - r^2}.\end{aligned}$$



Step-2:

$$x^2 + y^2 = 9, \quad y = x, \quad y = \sqrt{3}x.$$

$$\begin{aligned} 1) \quad x^2 + y^2 = 9 &\Rightarrow (r \cos \varphi)^2 + (r \sin \varphi)^2 = 9 \Rightarrow \\ &\Rightarrow r^2 (\cos^2 \varphi + \sin^2 \varphi) = 9 \Rightarrow \\ &\Rightarrow r^2 = 9 \Rightarrow r = 3; \end{aligned}$$

$$2) \quad y = x \Rightarrow r \sin \varphi = r \cos \varphi \Rightarrow \operatorname{tg} \varphi = 1 \Rightarrow \varphi = \frac{\pi}{4};$$

$$\begin{aligned} 3) \quad y = \sqrt{3}x &\Rightarrow r \sin \varphi = \sqrt{3}r \cos \varphi \Rightarrow \\ &\Rightarrow \operatorname{tg} \varphi = \sqrt{3} \Rightarrow \varphi = \frac{\pi}{3}. \end{aligned}$$

Step-3:

$$I = \iint_D \sqrt{9-r^2} r dr d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^3 r \sqrt{9-r^2} dr$$

$$\begin{aligned} I &= \varphi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} \bullet \left(-\frac{1}{2} \right) \int_0^3 (9-r^2)^{\frac{1}{2}} d(9-r^2) = \\ &= -\frac{1}{2} \left(\frac{\pi}{3} - \frac{\pi}{4} \right) \bullet \frac{2(9-r^2)^{\frac{3}{2}}}{3} \Big|_0^3 = \\ &= -\frac{1}{3} \bullet \frac{\pi}{12} (0 - 27) = \frac{3\pi}{4}. \end{aligned}$$

Рябушко, часть 3.

Самостоятельная работа

1. Вычислить $\iint_D (12 - x - y) dx dy$, если область D ограничена окружностью $x^2 + y^2 = 9$. (Ответ: 108л.)

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2. Вычислить $\iint_D (6 - 2x - 3y) dx dy$, если область D ограничена окружностью $x^2 + y^2 = 4$. (Ответ: 24л.)

3. Вычислить $\iint_D (4 - x - y) dx dy$, если область D ограничена окружностью $x^2 + y^2 = 2x$. (Ответ: 3л.)